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A polar theory for vibrations of thin elastic shells

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Abstract

In relation to a polar continuum, this paper presents a 2-D shear deformable theory for the high frequency vibrations of a thin elastic shell. To begin with, the 3-D fundamental equations of the micropolar elastic continuum are expressed as the Euler–Lagrange equations of a unified variational principle. Next, the kinematic variables of the shell are represented by the power series expansions in its thickness coordinate, and then, they are used to establish the 2-D theory by means of the variational principle. The 2-D theory is derived in invariant variational and differential forms and governs all the types of vibrations of the functionally graded micropolar shell. Lastly, the uniqueness is investigated in solutions of the initial mixed boundary value problems defined by the 2-D theory, and some of special cases are indicated in the theory.

Keywords: Functionally graded polar shells; High frequency vibrations; Uniqueness theorem

1. Introduction

Polar materials are certain class of materials possessing granular, fibrous or coarse-grain structure that can support the couple stresses and the body couples and are influenced by the spin inertia. Research interest in polar continua was initiated by Poisson in 1842 and was elaborated by Voigt in 1887 (see, e.g., Love, 1944), and many other pioneers, Kirchhoff, Clebsch, Duhem, the Cosserat brothers and Sudria made some contributions. The reason of departure from the classical (non-polar) theory of elasticity to the polar theory of elasticity was due to the discrepancies between the result of the classical theory and the experiments in certain instances. In fact, it is known that the classical theory fails to explain the dispersion phenomena at

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high frequencies and small wavelengths of surface waves. Besides, it is inadequate in treating the vibrations of granular materials and materials with large molecules such as polymers, and in analysing the stress concentrations near notches and cracks. The discrepancies were attributed to the basic axiom of the classical theory, that is, the transmission of interaction between any two particles of materials is assumed to be solely through the action of a force vector. As a first step to eliminate the discrepancies, Voigt (1887) postulated, contrary to Cauchy's second law of motion, that the transmission of interaction is through a force vector as well as a couple or moment vector, thus giving rise to the so-called couple stress theory of elasticity. In the couple stress or polar theory, a material element of continua has six-degrees of freedom, a displacement vector and an independent microrotation vector, whereas it has three-degrees of freedom in the non-polar theory. Following Voigt, the Cosserat brothers (1909) developed a rational polar theory of continua but omitting the constitutive relations. Some fifty years later, an explosion of renewed interest arose for polar continua and further emphasis was directed to related theories by a number of eminent scientists (e.g., Günther, 1958; Truesdell and Toupin, 1960; Grioli, 1960; Aero and Kuvshinskii, 1961; Mindlin and Tiersten, 1962; Toupin, 1964; Eringen and Şuhubi, 1964; Palmov, 1964; Koiter, 1964). The polar theories of continua with rigid and deformable directors were reviewed, including the historical background and developments, the inherent assumptions in the theories advanced to date, and the applications (e.g., Kunin, 1982, 1983; Capriz, 1989; Erofeyev, 2003).

Among the polar theories of continua, the micropolar theory that may be applicable to both solids and fluids is adopted here due to its deterministic, experimentally corroborated and hence universally accepted nature. Eringen and Şuhubi (1964) initiated the theory as a special case of their continuum theory of microelastic solids. Later, the theory was recapitulated, discussed its thermodynamical foundations and renamed it as the micropolar theory where the stretches and distortion of microstructure were excluded (i.e., rigid directors). The micropolar theory is now a subject of intensive study from both mathematical and engineering point of view. However, relatively few experimental studies of a corroborative character were conducted so as to identify the materials of which the behaviour tallies with that of micropolar continua (e.g., Gauthier, 1982). This is largely due to the difficulty in isolating the effect of couple stress, and hence, in determining the micropolar material constants. Recently, an up-to-date works devoted to micropolar elasticity theory were presented, including some static and dynamic applications (e.g., Dyszlewicz, 2004, and references therein).

The micropolar theory takes into account the granular character of a continuum and describes its deformations by a pair of the displacement and microrotation vectors. Of the fundamental equations of the theory, the divergence equations were developed in integral (global) from, and then, they were established in differential form under certain regularity and local differentiability conditions of the field variables. The rest of the fundamental equations (i.e., the gradient equations, the constitutive relations, and the boundary and initial conditions) were stated in differential form. The internal consistency of the fundamental equations, that is, the existence and uniqueness in their solutions was well established. The fundamental equations were alternatively expressed in variational form through some variational principles with their well-known features. Hellinger (1914) and much later Reissner (1950, 1965) was the first to obtain some variational principles that generate the divergence equations and the divergence equations together with the natural boundary conditions, respectively, in elasticity as well as in polar (oriented) elasticity. Several variational principles with their reciprocals were derived in micropolar elasticity, including even the non-local, thermal and electrical effects (e.g., Dökmeci, 1973, 1979; Nowacki, 1986; Felippa, 1992; Steinmann and Stein, 1997; Mosconi, 2002, and references cited therein). Most recently, as an extension of the Hellinger–Reissner variational principles, Altay and Dökmeci (2004a,b) formulated a unified variational principle for laminated polar elastic continua. This variational principle is now used in deriving the system of 2-D equations for the high frequency vibrations of micropolar shells.

Shells are the 3-D structural elements characterised by a geometric parameter $\epsilon \ll 1$, that is, their 1-D is much smaller than the others. Accordingly, they were mathematically modelled as a 2-D continuum and their physical response was almost always predicted by use of the 2-D equations deduced from the 3-D fundamental equations of continuum. In deducing the 2-D equations, different type of approaches based mainly on a kinematic hypothesis was employed, and they can be found in the treatises (Kil'chevskiy, 1965; Naghdi, 1972; Villaggio, 1997; Libai and Simmonds, 1998; Rubin, 2000). Both the 3-D fundamental equations and the 2-D equations inherently contain some inevitable as well as unmeasurable errors of experimental nature in the constitutive relations, though the former is more accurate and less tractable in computation than the latter. The relative merit of using either the fundamental equations or the lower order equations essentially depends on the geometric parameter; this point of importance was recently discussed (Altay and Dokmeci, 2003). By use of the 2-D equations, the vibrations of shells with various shapes, vibration modes and materials were abundantly investigated, including some experimental measurements of specific characteristics (e.g., Liew et al., 1993; Steele et al., 1995; Qatu, 2002). These works were mainly directed toward the vibrations of shells made of non-polar materials under the mechanical effects at low frequencies where the wavelength is large as compared with the shell thickness. At high frequencies where the wavelength is of the order of magnitude or smaller than the thickness, only a few works were reported dealing with the vibrations of shells subjected to the coupled mechanical and/or electrical, thermal and magnetic effects (e.g., Le, 1999; Wang and Yang, 2000; Altay and Dokmeci, 2001). However, no results were reported for the vibrations of shells made of functionally graded non-polar or micropolar materials. Functionally graded materials are properly conceived and tailored to achieve particular combination and characteristics of materials desired. They have properties varying continuously in certain direction as a new design feature for structural elements at high frequency vibrations in advanced technologies (Suresh and Mortensen, 1998).

Investigations concerning micropolar structural elements were rather scanty and mostly devoted to specific problems of rods (e.g., Smith, 1970; Potapenko et al., 2004), plates (e.g., Arıman, 1968) and shells with special geometrical configuration (Yeh and Chen, 1993). A 2-D theory of micropolar plates was formulated as an extension of the classical theory of elastic plates (Eringen, 1967) and using a method of asymptotic expansion (Green and Naghdi, 1967; Erbay, 2000). Some results under the plane stress assumption (Ambartsumian, 2002) and a canonical formulation (Saczuk, 1995) for the micropolar shells were given. Within the context of Love's second approximation of elastic shells, a 2-D theory of micropolar shells was deduced from the 3-D fundamental equations (Dökmeci, 1970). A comprehensive survey on shells made of polar materials was reported with no works on the vibrations of micropolar shells at high frequency or with functionally graded material and/or in fully variational form (Rubin, 2000; Qatu, 2002; Dyszlewicz, 2004; Rubin and Benveniste, 2004).

The aim of this paper is to derive a system of 2-D shear deformable equations for the high frequency vibrations of micropolar thin elastic shells in invariant differential and variational forms, to examine the uniqueness of solutions in the system of 2-D shell equations and to point out certain special cases.

The remainder of this section contains the notation to be used in the paper. This is followed in Section 2 by a summary of the 3-D fundamental equations of micropolar elasticity in differential form, and also in variational form. In Section 3, the kinematics of a micropolar shell is considered, the displacement and microrotation vectors are represented by the power series expansions in the thickness coordinate. Also, the 2-D linear gradient equations and non-linear constitutive relations of the functionally graded shell are recorded in this section. Section 4 is concerned with the derivation of the 2-D equations of the shell, including the divergence equations, the boundary and initial conditions by means of the unified variational principle together with the series expansions. A system of the 2-D equations of the shell is stated in both invariant differential and variational forms in Section 5. Next, the uniqueness in solutions of the initial mixed boundary value problems defined by the system of the 2-D shell equations is examined in Section 6. In the last section, some conclusions are drawn involving further needs of research, certain special cases and some applications. Some preliminaries of surface geometry were given for ease of reference in Appendix A, and nomenclature in Appendix B.

Notation. In this paper, standard space and surface tensors are freely used in an Euclidean 3-D space Ξ. Accordingly, Einstein's summation convention is implied for all repeated subscript or superscript indices, unless they are enclosed with parentheses. Latin indices with the range 1–3 are assigned to space tensors and Greek indices with the ranges 1 and 2 to surface tensors. The θ^i-system of the space Ξ is identified with a fixed, right-handed set of geodesic normal coordinates. A superposed dot stands for time differentiation, a comma for partial differentiation with respect to an indicated space coordinate, and a semicolon and a colon for covariant differentiation with respect to an indicated space coordinate using the space and surface metrics, respectively. Further, an asterisk is used to denote prescribed initial and boundary quantities. A finite and bounded, regular micropolar region $\Omega + \partial\Omega$ with its piecewise boundary surface $\partial\Omega$ and closure $\overline{\Omega}$ of the space is indicated by $\Omega(t)$ at time t. The time interval is denoted by $T = [t_0, t_1)$ where $t_1 > t_0$ may be infinity and the Cartesian product of the region Ω and the time interval T by $\Omega \times T$. An overbar is used to indicate the field quantities referred to the base vectors of the middle surface of a shell with its thickness $2h$, and $Z = [-h, h]$ to the thickness interval. Also, C_{mn} refers to a class of functions with derivatives of order up to and including (m) and (n) with respect to the space coordinates θ^i and time t. The (n) and (p) are used to indicate the non-polar and polar parts of a quantity, respectively.

2. Fundamental equations of micropolar elasticity

In the Euclidean 3-D space Ξ, consider a fixed, right-handed system of geodesic normal coordinates θ^i. Let $\Omega + \partial\Omega$ with its smooth boundary surface $\partial\Omega$ and closure $\overline{\Omega}$ denote the regular region of a micropolar continuum. The region is referred to the system of coordinates θ^i. The complementary regular sub-surfaces of the region $\Omega + \partial\Omega$ are indicated by $(\partial\Omega_u, \partial\Omega_t)$ and $(\partial\Omega_\phi, \partial\Omega_{\mathrm{m}})$, that is, $\partial\Omega_u \cup \partial\Omega_t = \partial\Omega$ and $\partial\Omega_u \cap \partial\Omega_t = \varnothing$, and so on, and the unit outward vector normal to the surfaces by n_i. Further, let $\overline{\Omega} \times T$ represent the domain of definitions for the micropolar elastic fields of functions of the coordinates θ^i and time t of the class $C_{\alpha\beta}$. The 3-D fundamental equations of the continuum may be grouped as the divergence and gradient equations, the constitutive relations and the initial and boundary conditions to supplement them. Now, for an ease of reference, they are summarised below for an anisotropic micropolar but non-local as well as non-relativistic elastic continuum (e.g., Eringen, 1999).

Divergence equations (Cauchy's first and second laws of motion)

$$D_n^j = t^{ij}_{;i} + \rho(f^j - a^j) = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.1}$$

$$D_p^j = m^{ij}_{;i} + \varepsilon^{jkl} t_{kl} + \rho(l^j - b^j) = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.2}$$

where t^{ij}, ρ, f^j, $a^i = \ddot{u}^j$, u_i; m^{ij}, l^j, $b^j = J^{ji}\ddot{\phi}_i$, J^{ij}, ϕ_i and ε^{ijk} stand for the asymmetric stress tensor, the mass density, the body force vector, the acceleration vector, the displacement vector; the couple stress tensor, the body couple vector, the microacceleration vector, the microinertia tensor, the microrotation vector, and the alternating tensor, respectively.

Gradient equations

$$G^n_{ij} = e_{ij} - (u_{j;i} + \varepsilon_{jik}\phi^k) = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.3}$$

$$G^p_{ij} = \epsilon_{ij} - \phi_{i;j} = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.4}$$

where e_{ij} and ϵ_{ij} are the strain and microstrain tensors.

Constitutive relations

$$C_n^{ij} = t^{ij} - \frac{\partial\Sigma}{\partial e_{ij}} = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.5}$$

$$C_p^{ij} = m^{ij} - \frac{\partial\Sigma}{\partial \epsilon_{ij}} = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.6}$$

where Σ denotes the elastic energy density. Its quadratic form is expressed by

$$\Sigma = \tfrac{1}{2}(C^{ijkl}e_{ij}e_{kl} + B^{ijkl}\epsilon_{ji}\epsilon_{lk} + 2A^{ijkl}e_{ij}\epsilon_{kl} - 2A^{ij}\Theta e_{ij} - 2B^{ij}\Theta\epsilon_{ij}) \tag{2.7}$$

By virtue of Eqs. (2.5) and (2.6), the linear constitutive equations are given by

$$C_n^{ij} = t^{ij} - (C^{ijkl}e_{kl} + A^{ijkl}\epsilon_{kl} - A^{ij}\Theta) = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.8}$$

$$C_p^{ij} = m^{ij} - (A^{klji}e_{kl} + B^{jikl}\epsilon_{kl} - B^{ji}\Theta) = 0 \quad \text{in } \overline{\Omega} \times T \tag{2.9}$$

In these equations, $\Theta(\theta^i)$ denotes a prescribed steady temperature increment, and (A^{ij}, B^{ij}) and $(C^{ijkl} = C^{klij}$, $B^{ijkl} = B^{klij})$ refer to the linear thermal expansion coefficients at constant stress and the isothermal elastic stiffnesses, respectively. Moreover, they are specialised to the case of isotropy as

$$A^{ij} = \beta_0 g^{ij}, \quad B^{ij} = 0, \quad A^{ijkl} = 0 \tag{2.10}$$

$$C^{ijkl} = \lambda g^{ij}g^{kl} + (\mu + \kappa)g^{ik}g^{jl} + \mu g^{il}g^{jk}, \quad B^{ijkl} = \alpha g^{ij}g^{kl} + \beta g^{il}g^{jk} + \gamma g^{ik}g^{jl} \tag{2.11}$$

Here, λ and μ stand for Lamé's elasticity constants, β_0 for the coefficient of linear thermal expansion, and α, β, γ and κ are the four additional elastic moduli of isotropic micropolar continuum. In view of Eqs. (2.10) and (2.11), the linear constitutive relations from Eqs. (2.8) and (2.9) takes the form

$$C_{nl}^{ij} = t^{ij} - \left[\lambda g^{kl}g^{ij}e_{kl} + (\mu + \kappa)g^{ik}g^{jl}e_{kl} + \mu g^{il}g^{jk}e_{lk} - B\Theta g^{ij}\right] = 0 \tag{2.12}$$

$$C_{pl}^{ij} = m^{ij} - (\alpha\epsilon_{kk}g^{ij} + \beta g^{ki}g^{lj}\epsilon_{kl} + \gamma\epsilon_{lk}g^{li}g^{kj}) = 0 \tag{2.13}$$

Also, the microinertia tensor becomes $J^{ij} = Jg^{ij}$.

Boundary conditions

$$B_n^j = t_*^j - n_i t^{ij} = 0 \quad \text{on } \partial\Omega_t \times T \tag{2.14}$$

$$B_i^n = u_i - u_i^* = 0 \quad \text{on } \partial\Omega_u \times T \tag{2.15}$$

$$B_p^j = m_*^j - n_i m^{ij} = 0 \quad \text{on } \partial\Omega_m \times T \tag{2.16}$$

$$B_i^p = \phi_i - \phi_i^* = 0 \quad \text{on } \partial\Omega_\phi \times T \tag{2.17}$$

where $t^j = n_i t^{ij}$ and $m^j = n_i m^{ij}$ are the stress and couple stress vectors, respectively.

Initial conditions

$$I_i^n = u_i(\theta^j, t_0) - v_i^*(\theta^j) = 0, \quad J_i^n = w_i^*(\theta^j) - \dot{u}_i(\theta^j, t_0) = 0 \quad \text{in } \Omega(t_0) \tag{2.18}$$

$$I_i^p = \phi_i(\theta^j, t_0) - \varphi_i^*(\theta^j) = 0, \quad J_i^p = \Phi_i^*(\theta^j) - \dot{\phi}_i(\theta^j, t_0) = 0 \quad \text{in } \Omega(t_0) \tag{2.19}$$

Fundamental equations. The system of foregoing Eqs. (2.1)–(2.9) is deterministic, that is, there exist 42 equations for 42 dependent variables, ($t^{ij} \in C_{10}$, $e_{ij} \in C_{00}$, $u_i \in C_{12}$) and ($m^{ij} \in C_{10}$, $\epsilon_{ij} \in C_{00}$, $\phi_i \in C_{12}$), of the functions of the space coordinates θ^i and time t. The boundary and initial conditions (2.14)–(2.19) are shown to be sufficient for a unique solution of the linear Eqs. (2.1)–(2.4), (2.8) and (2.9). Now the system of fundamental equations is expressed by the Euler–Lagrange equations of a unified variational principle (Altay and Dokmeci, 2004a,b) as follows:

Unified variational principle. The principle is deduced from Hamilton's principle by modifying it through an involutory transformation, and it is expressed as

$$\delta L\{\Lambda\} = 0 \tag{2.20}$$

with its functional of the form

$$L\{\Lambda\} = \int_T \mathrm{d}t \int_\Omega \big[k - \Sigma(e_{ij}, \epsilon_{ij})\big]\mathrm{d}V + \int_T \mathrm{d}t \int_\Omega (t^{ij}G^n_{ij} + m^{ij}G^p_{ij})\mathrm{d}V + \int_T \mathrm{d}t \int_\Omega \rho(f^i u_i + l^i\phi_i)\mathrm{d}V$$
$$+ \int_T \mathrm{d}t \int_{\partial\Omega_u} B^n_i n_j t^{ij}\,\mathrm{d}S + \int_T \mathrm{d}t \int_{\partial\Omega_t} t^i_* u_i\,\mathrm{d}S + \int_T \mathrm{d}t \int_{\Omega_\phi} B^p_i n_j m^{ij}\,\mathrm{d}S + \int_T \mathrm{d}t \int_{\partial\Omega_m} m^i_*\phi_i\,\mathrm{d}S \qquad (2.21a)$$

and its admissible state of the form

$$\Lambda_A = \{u_i \in C_{12}, e_{ij} \in C_{00}, t^{ij} \in C_{10};\ \phi_i \in C_{12}, \epsilon_{ij} \in C_{00}, m^{ij} \in C_{10}\} \qquad (2.21b)$$

for a regular region $\Omega + \partial\Omega$ of the micropolar continuum. In Eq. (2.20), the kinetic energy density is given by

$$k = k_n + k_p; \quad k_n = \tfrac{1}{2}\rho\dot{u}^i\dot{u}_i,\ \ k_p = \tfrac{1}{2}\rho J^{ij}\dot{\phi}_i\dot{\phi}_j \qquad (2.22)$$

Executing the variations indicated in Eq. (2.20), one arrives at the variational equation of the form

$$\delta L\{\Lambda\} = \delta L_n\{\Lambda_n\} + \delta L_p\{\Lambda_p\} = 0; \quad \Lambda_A = \Lambda_n \cup \Lambda_p \qquad (2.23)$$

Here, the denotations of the form

$$\delta L_n\{\Lambda_n = u_i, e_{ij}, t^{ij}\} = \int_T \mathrm{d}t \int_\Omega (D^i_n\delta u_i + G^n_{ij}\delta t^{ij} + C^{ij}_n\delta e_{ij})\mathrm{d}V + \int_T \mathrm{d}t \int_{\partial\Omega_t} B^i_n\delta u_i\,\mathrm{d}S$$
$$+ \int_T \mathrm{d}t \int_{\partial\Omega_u} B^n_i n_j\delta t^{ij}\,\mathrm{d}S \qquad (2.24)$$

$$\delta L_p\{\Lambda_p = \phi_i, \epsilon_{ij}, m^{ij}\} = \int_T \mathrm{d}t \int_\Omega (D^i_p\delta\phi_i + G^p_{ij}\delta m^{ij} + C^{ij}_p\delta\epsilon_{ij})\mathrm{d}V + \int_T \mathrm{d}t \int_{\partial\Omega_m} B^i_p\delta\phi_i\,\mathrm{d}S$$
$$+ \int_T \mathrm{d}t \int_{\partial\Omega_\phi} B^p_i n_j\delta m^{ij}\,\mathrm{d}S \qquad (2.25)$$

are introduced.

Evidently, the Euler–Lagrange equations of the functional have all the fundamental equations of micropolar elasticity except the initial conditions. The variational principle with its functional is an integral type of variational principles with their well-known features. The functional (2.21) contains the virtual work associated with the surface tractions and couples, t^i_* and m^i_*, and hence it is expressed in a mathematically, though not physically, an appropriate form.

3. Deformation and strain fields, and constitutive relations

General background. The shell of uniform thickness $2h$ is mathematically approximated to a 2-D surface by the fundamental assumption of the form

$$\epsilon_s = \frac{2h}{R_0} \ll 1, \quad \eta_s = \frac{2h}{L} \ll 1 \qquad (3.1a)$$

Here, ϵ_s and η_s, R_0 and L represent the shell parameters, the least principal radius of curvature and the smallest structural dimension of the middle surface, respectively. Besides, the first of the restriction (3.1) is a sufficient condition to ensure the existence of the shell tensor (or the shifters) μ^α_β that plays an important role in the relationships between the space and surface tensors (Naghdi, 1972).

Moreover, no singularity or discontinuity of geometric and loading type is considered in the shell region $V + \partial V$, and hence, all the field variables together with their derivatives are taken to be exist, single-valued and piecewise continuous functions of the space coordinates and time. The field variables are assumed not to vary widely across the thickness interval $Z = [-h, h]$ of the shell, and hence, to be averaged over the interval Z. Besides, the shell region is taken to be at an elastic range under the mechanical and steady thermal loads. The micropolar region, $V + \partial V$, is treated as a thin (not moderately thick or thick) shell within the linear theory of micropolar elasticity. The thinness hypothesis is characterised by the parameters ϵ_s and η_s, while the range of applicability of the geometrically linear theory is primarily determined by another shell parameter of the form

$$\lambda_s = \frac{\max|u_i|}{2h} \tag{3.1b}$$

where u_i are the components of the mechanical displacements of the shell. Only, a very few authors (e.g., Kil'chevskiy, 1965; John, 1965; Berger, 1973) were concerned with the upper limits of the shell parameters, and they proposed somewhat arbitrary limits based on the physical considerations. The parameters evidently appeal to much greater elaboration that involves simultaneously the errors of both the kinematical and constitutive nature as discussed in a treatise (Libai and Simmonds, 1998) and a recent comment (Altay and Dökmeci, 2003).

Method of reduction. Among the various methods of reduction (e.g., Koiter and Simmonds, 1973; Gol'denveizer, 1997; Pikul, 2000), the method of series expansions is used for the derivation of the 2-D equations of a micropolar shell. This method rests entirely upon the choice of a field variable as a basis of the derivation at the outset, that is, its series expansions in the thickness coordinate that should be complete, and then, on a direct integration or variational averaging procedure over the thickness interval. The deformation field is chosen as the basis in deducing the 2-D shell equations from the 3-D fundamental equations of micropolar elasticity. The kinematical choice is a conventional one due to the fact that the differentiation operation is usually simpler than the integration operation, and also, no further equations of compatibility type involve with the derivation. A large class of refined theories of shells (plates) and rods made of several materials was derived by use of kinematic hypotheses. Besides, all the significant effects of higher orders can be taken into account by use of the series expansions of the deformation field. The series expansions can be truncated at a specific order of approximation so as to yield the classical results of elastic shells and plates. Moreover, any other field quantity may be selected as the basis of derivation in lieu of the deformation field (e.g., stresses, strains, and energy). The choice of the basis is an important one, and it was recently discussed, including various methods of reduction (Altay and Dökmeci, 2003).

Micropolar deformation field. In view of the fundamental assumptions, (3.1), the shifted components of the displacement and microrotation vectors of the micropolar shell are approximately expressed by

$$\bar{u}_i(\theta^i, t) = \sum_{n=0}^{M} u_i^{(n)}(\theta^\alpha, t)\Psi_{(n)}^{(i)}(\theta^3) \tag{3.2}$$

$$\bar{\phi}_i(\theta^i, t) = \sum_{n=0}^{N} \phi_i^{(n)}(\theta^\alpha, t)\Phi_{(n)}^{(i)}(\theta^3) \tag{3.3}$$

Here, the shifted components stand for the components referred to the base vectors of the middle surface. In Eqs. (3.2) and (3.3), from the mathematical standpoint, a separation of variables solution is sought for the 2-D equations of the micropolar shell which are derived in the following sections. Thus, the vector functions, $\bar{u}_i^{(n)} \in C_{12}$ and $\bar{\phi}_i^{(n)} \in C_{12}$, of order (n) are unknown a priori and independent functions of the surface coordinates θ^α and time t. The approximating functions, $\Psi_i^{(n)}$ and $\Phi_i^{(n)}$, are consistently taken in the form

$$\Psi_i^{(n)} = \Phi_i^{(n)} = z^n; \quad M = N \tag{3.4}$$

The kinematic hypothesis, Eqs. (3.2)–(3.4), is a generalisation of the Kirchhoff–Love hypotheses of elastic shells that was widely used in deriving the higher order theories of structural elements (Wang and Yang, 2000). The conventional shell equations include only the two tangential components of the microrotation vector, that is, ϕ_α, while the third component is also retained in Eq. (3.2) for ease of derivation, and it can be readily omitted in the resulting 2-D shell equations.

Micropolar gradient equations. The kinematic expansions, (3.2)–(3.4), obviously imply similar expansions for their derivatives in the gradient Eqs. (2.3) and (2.4), namely

$$\{e_{ij}(\theta^i,t),\epsilon_{ij}(\theta^i,t)\} = \sum_{n=0}^{N} z^n \{e_{ij}^{(n)}(\theta^\alpha,t),\epsilon_{ij}^{(n)}(\theta^\alpha,t)\} \tag{3.5}$$

To obtain the explicit expressions of the strain and microstrain of order (n), the variational gradient equation from Eq. (2.23) together with Eqs. (2.3) and (2.4) is stated by

$$\delta L_g\{t^{ij},m^{ij}\} = \delta L_{gn}\{t^{ij}\} + \delta L_{gp}\{m^{ij}\} \tag{3.6}$$

with

$$\delta L_{gn}\{t^{ij}\} = \int_T \mathrm{d}t \int_Z \mathrm{d}z \int_A \mu[e_{ij} - (u_{j;i} + \varepsilon_{jik}\phi^k)]\delta t^{ij}\,\mathrm{d}A \tag{3.7a}$$

$$\delta L_{gp}\{m^{ij}\} = \int_T \mathrm{d}t \int_Z \mathrm{d}z \int_A \mu(\epsilon_{ij} - \phi_{j;i})\delta m^{ij}\,\mathrm{d}A \tag{3.7b}$$

for the shell region. The expansions (3.5) are inserted into these equations by replacing the deformation components (3.2) and (3.3), and utilizing the relations (A.11), the components and their covariant derivatives are expressed in terms of their shifted equivalents and the derivatives with respect to the surface metrics, respectively. Then, the integration is performed with respect to the thickness coordinate, and the strain and microstrain fields of order (n) are obtained as

$$\delta L_g\{T_{(n)}^{ij},M_{(n)}^{ij}\} = \int_T \mathrm{d}t \int_A \sum_{n=0}^{N}\left[(e_{ij}^{(n)} - \varsigma_{ij}^{(n)})\delta T_{(n)}^{ij} + (\epsilon_{ij}^{(n)} - \zeta_{ij)}^{(n)})\delta M_{(n)}^{ij}\right]\mathrm{d}A \tag{3.8}$$

Here, $\varsigma_{(n)}^{ij}(\theta^\alpha,t)$ and $\zeta_{ij}^{(n)}(\theta^\alpha,t)$ are defined by

$$\begin{aligned}
\varsigma_{\alpha\beta}^{(n)} &= u_{\beta:\alpha}^{(n)} - b_{\beta\alpha}u_3^{(n)} - (b_\beta^v u_{v:\ \alpha}^{(n-1)} - c_{\alpha\beta}u_3^{(n-1)}) + \bar{\varepsilon}_{\beta\alpha}(\phi_3^{(n)} - 2R_\mathrm{m}\phi_3^{(n-1)} + R_\mathrm{g}\phi_3^{(n-2)})\\
\varsigma_{\alpha 3}^{(n)} &= u_{3,\alpha}^{(n)} + b_\alpha^\beta u_\beta^{(n)} + \bar{\varepsilon}_{\alpha\beta}[\phi_{(n)}^\beta + (b_v^\beta - 2R_\mathrm{m}\delta_v^\beta)\phi_{(n-1)}^v]\\
\varsigma_{3\alpha}^{(n)} &= (n+1)u_\alpha^{(n+1)} - nb_\alpha^\beta u_\beta^{(n)} + \bar{\varepsilon}_{\alpha\beta}[\phi_{(n)}^\beta + (b_v^\beta - 2R_\mathrm{m}\delta_v^\beta)\phi_{(n-1)}^v]; \quad \varsigma_{33}^{(n)} = (n+1)u_3^{(n+1)}
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\zeta_{\alpha\beta}^{(n)} &= \phi_{\beta:\alpha}^{(n)} - b_{\beta\alpha}\phi_3^{(n)} - (b_\beta^v\phi_{v:\ \alpha}^{(n-1)} - c_{\alpha\beta}\phi_3^{(n-1)}); \quad \zeta_{\alpha 3}^{(n)} = \phi_{3,\alpha}^{(n)} + b_\alpha^\beta\phi_\beta^{(n)}\\
\zeta_{3\alpha}^{(n)} &= (n+1)\phi_\alpha^{(n+1)} - nb_\alpha^\beta\phi_\beta^{(n)}; \quad \varsigma_{33}^{(n)} = (n+1)\phi_3^{(n+1)}
\end{aligned} \tag{3.10}$$

Besides, the stress and couple stress resultants of order (n) that may be readily reduced to those defined in terms of the physical components through

$$\{T_{(n)}^{ij},M_{(n)}^{ij}\} = \int_Z \{t^{ij},m^{ij}\}\mu z^n\,\mathrm{d}z \tag{3.11}$$

per unit length of the coordinate curves on the reference surface are introduced in Eq. (3.8).

Constitutive relations. The material of the micropolar shell is taken to be functionally graded along the thickness, and hence, the elastic energy density Σ, the mass density ρ and the material coefficients, A^{ijkl}, B^{ijkl}, C^{ijkl}, A^{ij} and B^{ij} in Eqs. (2.5) and (2.6) are continuously dependent on the thickness coordinate z. The dependency on the thickness coordinate was discussed and several criteria were proposed in choosing it (e.g., Markworth et al., 1995). Keeping in mind the dependency of the material coefficients, consider the variational constitutive equation from Eq. (2.23) together with Eqs. (2.5) and (2.6), namely

$$\delta L_c\{e_{ij},\epsilon_{ij}\} = \delta L_{cn}\{e_{ij}\} + \delta L_{cp}\{\epsilon_{ij}\} \tag{3.12a}$$

with

$$\begin{aligned}\delta L_{cn}\{e_{ij}\} &= \int_T \mathrm{d}t \int_Z \mathrm{d}z \int_A \left(t^{ij} - \frac{\partial\Sigma}{\partial e_{ij}}\right)\delta e_{ij}\mu\,\mathrm{d}A,\\ \delta L_{cp}\{\epsilon_{ij}\} &= \int_T \mathrm{d}t \int_Z \mathrm{d}z \int_A \left(m^{ij} - \frac{\partial\Sigma}{\partial \epsilon_{ij}}\right)\delta\epsilon_{ij}\mu\,\mathrm{d}A\end{aligned} \tag{3.12b}$$

for the shell region. Substituting Eq. (3.6) into this equation, and then integrating with respect to the thickness coordinate z and using Eq. (3.11), one reads the variational constitutive relations of the form

$$\begin{aligned}\delta L_{cn}\{e^{(n)}_{ij}\} &= \int_T \mathrm{d}t \int_A \left(T^{ij}_{(n)} - \frac{\partial\Gamma}{\partial e^{(n)}_{ij}}\right)\delta e^{(n)}_{ij}\,\mathrm{d}A,\\ \delta L_{cp}\{\epsilon^{(n)}_{ij}\} &= \int_T \mathrm{d}t \int_A \left(M^{ij}_{(n)} - \frac{\partial\Gamma}{\partial \epsilon^{(n)}_{ij}}\right)\delta\epsilon^{(n)}_{ij}\,\mathrm{d}A\end{aligned} \tag{3.13}$$

Here, the elastic energy per unit area of the reference surface A by

$$\Gamma = \int_Z \Sigma\mu\,\mathrm{d}z \tag{3.14}$$

is defined.

In order to obtain the linear constitutive equations of the micropolar shell, the quadratic version of the elastic energy function in Eq. (2.7) is inserted into Eq. (4.14) and then it is evaluated as before with the result

$$\begin{aligned}\delta L_{cn}\{e^{(n)}_{ij}\} &= \int_T \mathrm{d}t \int_A \left(T^{ij}_{(n)} - T^{ij}_{(n)c}\right)\delta e^{(n)}_{ij}\mathrm{d}A\\ \delta L_{cp}\{\epsilon^{(n)}_{ij}\} &= \int_T \mathrm{d}t \int_A \left(M^{ij}_{(n)} - M^{ij}_{(n)c}\right)\delta\epsilon^{(n)}_{ij}\mathrm{d}A\end{aligned} \tag{3.15}$$

where

$$T^{ij}_{(n)c} = \sum_{m=0}^{N}\left(C^{ijkl}_{(m+n)}e^{(m)}_{ij} + A^{ijkl}_{(m+n)}\epsilon^{(m)}_{kl} - A^{ij}_{(m+n)}\Theta_{(m)}\right) \tag{3.16}$$

$$M^{ij}_{(n)c} = \sum_{m=0}^{N}\left(B^{ijkl}_{(m+n)}\epsilon^{(m)}_{ij} + A^{ijkl}_{(m+n)}e^{(m)}_{kl} - B^{ij}_{(m+n)}\Theta_{(m+n)}\right) \tag{3.17}$$

Here, the prescribed steady temperature increment is represented by the power series in the thickness coordinate of the from

$$\Theta = \sum_{n=0}^{N}\Theta_{(n)}z^n \tag{3.18}$$

in consistent with the deformation field, Eqs. (3.2) and (3.3). Also, the material coefficients of order (n) by

$$\left\{A^{ijkl}_{(n)}(\theta^\alpha), B^{ijkl}_{(n)}(\theta^\alpha), C^{ijkl}_{(n)}(\theta^\alpha), A^{ij}_{(n)}, B^{ij}_{(n)}(\theta^\alpha)\right\} = \int_Z \{A^{ijkl}, B^{ijkl}, C^{ijkl}, A^{ij}, B^{ij}\}z^n\mu\,\mathrm{d}z \tag{3.19}$$

are introduced. In the case of an homogeneous shell material, the coefficients take the form

$$\{A^{ijkl}_{(n)}(\theta^\alpha), B^{ijkl}_{(n)}(\theta^\alpha), C^{ijkl}_{(n)}(\theta^\alpha), A^{ij}_{(n)}(\theta^\alpha), B^{ij}_{(n)}(\theta^\alpha)\} = \mu_n\{A^{ijkl}, B^{ijkl}, C^{ijkl}, A^{ij}, B^{ij}\} \tag{3.20}$$

where

$$\{I_n, \mu_n\} = \int_Z z^n\{1, \mu\}\,\mathrm{d}z; \quad \mu_n = I_n - 2I_{(n+1)}R_\mathrm{m} + I_{(n+2)}R_\mathrm{g}; \quad I_{n=2p} = \frac{2h^{n+1}}{(n+1)}, \quad I_{n=2p+1} = 0 \tag{3.21}$$

Here, I_n is the moment of inertia of order (n).

4. Divergence equations and boundary and initial conditions

In this section, paralleling to the derivation of the 2-D shell equations in the previous section, the 2-D divergence equations and the boundary conditions of the shell are deduced in variational form from the 3-D fundamental equations of micropolar elasticity. The derivation of the 2-D equations can be also accomplished through a direct integration of the 3-D fundamental equations in differential form. This will not be included here, since the differential shell equations are reported in the next section.

Micropolar divergence equations. With the aid of Eqs. (2.1), (2.2) and (2.23), the variational divergence equation is written as

$$\delta L_d\{\bar{u}_i, \bar{\phi}_i\} = \delta L_{dn}\{\bar{u}_i\} + \delta L_{dp}\{\bar{\phi}_i\} \tag{4.1}$$

with the denotations of the form

$$\begin{aligned}
\delta L_{dn}\{\bar{u}_i\} &= \int_T \mathrm{d}t \int_Z \mathrm{d}z \int_A \left\{ \left[t^{\beta\sigma}_{;\beta} + t^{3\sigma}_{;3} + \rho(f^\sigma - a^\sigma) \right] \mu^\alpha_\sigma \delta\bar{u}_\alpha + \left[t^{\alpha 3}_{;\alpha} + t^{33}_{;3} + \rho(f^3 - a^3) \right] \delta\bar{u}_3 \right\} \mu\,\mathrm{d}A \\
\delta L_{dp}\{\bar{\phi}_i\} &= \int_T \mathrm{d}t \int_Z \mathrm{d}z \int_A \left\{ \left[m^{\beta\sigma}_{;\beta} + m^{3\sigma}_{;3} + g^{\beta\sigma}\varepsilon_{\beta kl}t^{kl} + \rho(l^\sigma - b^\sigma) \right] \mu^\alpha_\sigma \delta\bar{u}_\alpha \right. \\
&\quad \left. + \left[m^{\alpha 3}_{;\alpha} + m^{33}_{;3} + \varepsilon_{3kl}t^{kl} + \rho(l^3 - b^3) \right] \delta\bar{\phi}_3 \right\} \mu\,\mathrm{d}A
\end{aligned} \tag{4.2}$$

in terms of the shifted components of the deformation field. Inserting the expansions of the deformation field, (3.2) and (3.3), into these equations, replacing the covariant derivatives with respect to the space metrics by those designated by a colon (:) with respect to the surface metrics through the identities (A.12)–(A.14) and using (A.11), the variational equations above may be written as

$$\begin{aligned}
\delta L_{dn}\{u^{(n)}_i\} &= \int_T \mathrm{d}t \int_A \mathrm{d}A \int_Z \mu \mathrm{d}z \sum_{n=0}^{N} \left\langle \left\{ [\mu(t^{\beta\alpha} - zb^\alpha_\sigma t^{\beta\sigma})z^n]_{:\,\beta}. - \mu b^\alpha_\beta t^{\beta 3} z^n - n\mu(t^{3\alpha} - zb^\alpha_\sigma t^{3\sigma})z^{n-1} \right. \right. \\
&\quad \left. + \left[\mu(t^{3\alpha} - zb^\alpha_\beta t^{3\beta})z^n \right]_{,3} + \rho(\mu^\alpha_\beta f^\beta - \ddot{u}^\alpha) \right\} \delta u^{(n)}_\alpha \\
&\quad \left. + [(\mu t^{\alpha 3} z^n)_{:\,\alpha} + \mu b_{\sigma\beta}(t^{\beta\sigma} - zb^\sigma_\alpha t^{\beta\alpha})z^n - n\mu t^{33} z^{n-1} + \rho(f^3 - \ddot{u}^3) + (\mu t^{33} z^n)_{,3}] \delta u^{(n)}_3 \right\rangle
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\delta L_{dp}\{\phi^{(n)}_i\} &= \int_T \mathrm{d}t \int_A \mathrm{d}A \int_Z \mu \mathrm{d}z \sum_{n=0}^{N} \left\langle \left\{ [\mu(m^{\beta\alpha} - zb^\alpha_\sigma m^{\beta\sigma})z^n]_{:\,\beta}. - \mu b^\alpha_\beta m^{\beta 3} z^n - n\mu(m^{3\alpha} - zb^\alpha_\sigma m^{3\sigma})z^{n-1} \right. \right. \\
&\quad \left. + \left[\mu(m^{3\alpha} - zb^\alpha_\beta m^{3\beta})z^n \right]_{,3} + \bar{\varepsilon}_{\beta\sigma}(a^{\alpha\beta} + zd^{\beta\alpha})t^{\sigma[3]} + \rho(\mu^\alpha_\beta l^\beta - \ddot{\phi}^\alpha) \right\} \delta\phi^{(n)}_\alpha \\
&\quad + \left[(\mu m^{\alpha 3} z^n)_{:\,\alpha} + \mu b_{\sigma\beta}(m^{\beta\sigma} - zb^\sigma_\alpha m^{\beta\alpha})z^n - n\mu m^{33} z^{n-1} \right. \\
&\quad \left. \left. + \mu\bar{\varepsilon}_{\alpha\beta}t^{\alpha\beta} + \rho(l^3 - \ddot{\phi}^3) + (\mu m^{33} z^n)_{,3} \right] \delta\phi^{(n)}_3 \right\rangle
\end{aligned} \tag{4.4}$$

Here, some simplification together with Eq. (A.14) is considered.

After applying Green's theorem, and then using Eq. (A.6) and integrating with respect to the thickness coordinate, one finally obtains the variational divergence equations of the micropolar shell of the form

$$\delta L_d\{u_i^{(n)}, \phi_i^{(n)}\} = \delta L_{dn}\{u_i^{(n)}\} + \delta L_{dp}\{\phi_i^{(n)}\} \tag{4.5a}$$

with

$$\begin{aligned} \delta L_{dn}\{u_i^{(n)}\} &= \int_T \mathrm{d}t \int_A \sum_{n=0}^{N} (V^i_{(n)} + F^i_{(n)} - \ddot{A}^i_{(n)})\delta u_i^{(n)}\,\mathrm{d}A \\ \delta L_{dp}\{\phi_i^{(n)}\} &= \int_T \mathrm{d}t \int_A \sum_{n=0}^{N} (W^i_{(n)} + L^i_{(n)} - \ddot{B}^i_{(n)})\delta \phi_i^{(n)}\,\mathrm{d}A \end{aligned} \tag{4.5b}$$

Here, the denotations of the form

$$\begin{aligned} V^\alpha_{(n)} &= \left(T^{\beta\alpha}_{(n)} - b^\alpha_\sigma T^{\beta\sigma}_{(n+1)}\right)_{:\,\beta} - b^\alpha_\beta T^{\beta 3}_{(n)} - n(T^{3\alpha}_{(n-1)} - b^\alpha_\beta T^{3\beta}_{(n)}) + P^\alpha_{n(n)} - Q^\alpha_{n(n)} \\ V^3_{(n)} &= T^{\alpha 3}_{(n):\,\alpha} + b_{\alpha\beta}(T^{\beta\alpha}_{(n)} - b^\alpha_\sigma T^{\beta\sigma}_{(n+1)}) - nT^{33}_{(n-1)} + P^3_{n(n)} - Q^3_{n(n)} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} W^\alpha_{(n)} &= (M^{\beta\alpha}_{(n)} - b^\alpha_\sigma M^{\beta\sigma}_{(n+1)})_{:\,\beta} - b^\alpha_\beta M^{\beta 3}_{(n)} - n(M^{3\alpha}_{(n-1)} - b^\alpha_\beta M^{3\beta}_{(n)}) + P^\alpha_{p(n)} - Q^\alpha_{p(n)} \\ &\quad + \bar{\varepsilon}_{\beta\sigma}\left(a^{\alpha\beta} T^{\sigma[3]}_{(n)} + d^{\beta\alpha} T^{\sigma[3]}_{(n+1)}\right) \\ W^3_{(n)} &= M^{\alpha 3}_{(n):\,\alpha} + b_{\alpha\beta}(M^{\beta\sigma}_{(n)} - b^\alpha_\sigma M^{\beta\sigma}_{(n+1)}) - nM^{33}_{(n-1)} + P^3_{p(n)} - Q^3_{p(n)} \\ &\quad + \bar{\varepsilon}_{\alpha\beta}\left(T^{\alpha\beta}_{(n)} - b^\sigma_\sigma T^{\alpha\beta}_{(n+1)} + R_{\mathrm{g}} T^{\alpha\beta}_{(n+2)}\right) \end{aligned} \tag{4.7}$$

are given in terms of the stress and couple stress resultants (3.11). The surface loads of order (n), measured per unit area of the middle surface, by

$$\begin{aligned} \{P^i_{n(n)}, Q^i_{n(n)}\} &= \mu(t^{3i} - zb^\alpha_\beta t^{3\beta}\delta^i_\alpha)z^n \quad \text{at } z = \{h, -h\} \\ \{P^i_{p(n)}, Q^i_{p(n)}\} &= \mu(m^{3i} - zb^\alpha_\beta m^{3\beta}\delta^i_\alpha)z^n \quad \text{at } z = \{h, -h\} \end{aligned} \tag{4.8}$$

and the body force and body couple resultants of order (n) by

$$F^i_{(n)} = \int_Z \rho\mu z^n (f^i - zb^\alpha_\beta f^\beta \delta^i_\alpha)\mathrm{d}z; \quad L^i_{(n)} = \int_Z \rho\mu z^n (l^i - zb^\alpha_\beta l^\beta \delta^i_\alpha)\mathrm{d}z \tag{4.9}$$

per unit area of A are given. The displacement and microrotation resultants of order (n) in the form

$$\{A^i_{(n)}, B^i_{(n)}\} = \sum_{m=0}^{N} \rho_{(m+n)}\{u^i_{(m)}, J^{ij}\phi_j^{(m)}\}; \quad \rho_{(n)} = \int_Z \rho\mu z^n\,\mathrm{d}z \tag{4.10}$$

are introduced, and also, the notation as

$$t^{\alpha[3]} = t^{\alpha 3} - t^{3\alpha} \tag{4.11}$$

is defined for an asymmetric stress tensor. In Eqs. (4.5) and (4.9), the mass density is taken to be $\rho = \rho(z)$ in consistent with the dependency of all the material properties.

Natural boundary conditions. From Eq. (2.23) and Eqs. (2.14)–(2.17), the variational boundary conditions are expressed as

$$\delta L_b = \delta L_{bn}\{u_i, t^{ij}\} + \delta L_{bp}\{\phi_i, m^{ij}\} \tag{4.12}$$

with

$$\delta L_{bn}\{u_i, t^{ij}\} = \int_T \mathrm{d}t \int_{\partial\Omega_t} (t_*^j - n_i t^{ij})\delta u_i \, \mathrm{d}S + \int_T \mathrm{d}t \int_{\partial\Omega_u} (u_i - u_i^*) n_j \delta t^{ij} \, \mathrm{d}S \tag{4.13}$$

$$\delta L_{bp}\{\phi_i, m^{ij}\} = \int_T \mathrm{d}t \int_{\partial\Omega_m} (m_*^j - n_i m^{ij})\delta\phi_i \, \mathrm{d}S + \int_T \mathrm{d}t \int_{\partial\Omega_u} (\phi_i - \phi_i^*) n_j \delta m^{ij} \, \mathrm{d}S \tag{4.14}$$

The deformation field is taken to be prescribed on certain part of the edge boundary surface and the tractions are given on the remaining part of the edge boundary surface and the faces of the shell, namely

$$\partial\Omega_t = C_t \cup S_{\mathrm{lf}} \cup S_{\mathrm{uf}}, \;\; \partial\Omega_u = C_u; \quad \partial\Omega_m = C_m \cup S_{\mathrm{lf}} \cup S_{\mathrm{uf}}, \;\; \partial\Omega_\phi = C_\phi \tag{4.15}$$

Thus, the variational boundary condition (4.12) reads

$$\begin{aligned} \delta L_{bn} = & \int_T \mathrm{d}t \int_Z \mathrm{d}z \int_{C_t} \left[(t_*^\beta - v_\sigma t^{\sigma\beta})\mu_\beta^\alpha \delta\bar{u}_\alpha + (t_*^3 - v_\alpha t^{\alpha 3})\delta\bar{u}_3 \right] \mu \, \mathrm{d}c \\ & + \int_T \mathrm{d}t \int_{S_f} \left[(t_*^\beta - n_3 t^{3\beta})\mu_\beta^\alpha \delta\bar{u}_\alpha + (t_*^3 - n_3 t^{33})\delta\bar{u}_3 \right] \mathrm{d}S \\ & + \int_T \mathrm{d}t \int_Z \mathrm{d}z \int_{C_u} v_\alpha (u_i - u_i^*) \delta t^{\alpha i} \mu \, \mathrm{d}c \end{aligned} \tag{4.16}$$

$$\begin{aligned} \delta L_{bp} = & \int_T \mathrm{d}t \int_Z \mathrm{d}z \int_{C_m} \left[(m_*^\beta - v_\sigma m^{\sigma\beta})\mu_\beta^\alpha \delta\bar{\phi}_\alpha + (m_*^3 - v_\alpha m^{\alpha 3})\delta\bar{\phi}_3 \right] \mu \, \mathrm{d}c \\ & + \int_T \mathrm{d}t \int_{S_f} \left[(m_*^\beta - n_3 m^{3\beta})\mu_\beta^\alpha \delta\bar{\phi}_\alpha + (m_*^3 - n_3 m^{33})\delta\bar{\phi}_3 \right] \mathrm{d}S \\ & + \int_T \mathrm{d}t \int_Z \mathrm{d}z \int_{C_\phi} v_\alpha (\phi_i - \phi_i^*) \delta m^{\alpha i} \mu \, \mathrm{d}c \end{aligned} \tag{4.17}$$

where Eq. (A.10) is considered.

Inserting the expansions (3.2) and (3.3) into the equations above and evaluating as before, the boundary conditions are obtained as

$$\begin{aligned} \delta L_{bn} = & \int_T \mathrm{d}t \int_{C_t} \sum_{n=0}^{N} \{ [T_{*(n)}^\alpha - v_\sigma (T_{(n)}^{\sigma\alpha} - b_\beta^\alpha T_{(n+1)}^{\sigma\beta})]\delta u_\alpha^{(n)} + (T_{*(n)}^3 - v_\alpha T_{(n+1)}^{\alpha 3})\delta u_3^{(n)} \} \mathrm{d}c \\ & + \int_T \mathrm{d}t \int_{S_{\mathrm{uf}}} \sum_{n=0}^{N} (P_{*n(n)}^i - P_{n(n)}^i)\delta u_i^{(n)} \mathrm{d}A + \int_T \mathrm{d}t \int_{S_{\mathrm{lf}}} \sum_{n=0}^{N} (Q_{*n(n)}^i + Q_{n(n)}^i)\delta u_i^{(n)} \, \mathrm{d}A \\ & + \int_T \mathrm{d}t \int_{C_u} \sum_{n=0}^{N} v_\alpha (u_i^{(n)} - u_i^{*(n)}) \delta T_{b(n)}^{\alpha i} \, \mathrm{d}c \end{aligned} \tag{4.18}$$

$$\begin{aligned} \delta L_{bp} = & \int_T \mathrm{d}t \int_{C_m} \sum_{n=0}^{N} \{ [M_{*(n)}^\alpha - v_\sigma (M_{(n)}^{\sigma\alpha} - b_\beta^\alpha M_{(n+1)}^{\sigma\beta})]\delta\phi_\alpha^{(n)} + (M_{*(n)}^3 - v_\alpha M_{(n+1)}^{\alpha 3})\delta\phi_3^{(n)} \} \mathrm{d}c \\ & + \int_T \mathrm{d}t \int_{S_{\mathrm{uf}}} \sum_{n=0}^{N} (P_{*p(n)}^i - P_{p(n)}^i)\delta\phi_i^{(n)} \, \mathrm{d}A + \int_T \mathrm{d}t \int_{S_{\mathrm{lf}}} \sum_{n=0}^{N} (Q_{*p(n)}^i + Q_{p(n)}^i)\delta\phi_{(n)}^i \, \mathrm{d}A \\ & + \int_T \mathrm{d}t \int_{C_\phi} \sum_{n=0}^{N} v_\alpha (\phi_i^{(n)} - \phi_i^{*(n)}) \delta M_{b(n)}^{\alpha i} \, \mathrm{d}c \end{aligned} \tag{4.19}$$

Here, in consistent with Eqs. (3.2) and (3.3), the prescribed deformation field is expressed by the power series expansions in the thickness coordinate. Also, the edge and surface loads of order (n) by

$$T^i_{*(n)} = \int_Z (t^i_* - zb^\alpha_\beta t^\beta_* \delta^i_\alpha)\mu z^n \,\mathrm{d}z; \quad \{P^i_{*n(n)}, Q^i_{*n(n)}\} = (t^i_* - zb^\alpha_\beta t^\beta_* \delta^i_\alpha)\mu z^n \quad \text{at } z = \{h, -h\} \tag{4.20a}$$

and

$$\begin{aligned} M^i_{*(n)} &= \int_Z (m^i_* - zb^\alpha_\beta m^\beta_* \delta^i_\alpha)\mu z^n \,\mathrm{d}z; \\ \{P^i_{*p(n)}, Q^i_{*p(n)}\} &= (m^i_* - zb^\alpha_\beta m^\beta_* \delta^i_\alpha)\mu z^n \quad \text{at } z = \{h, -h\} \end{aligned} \tag{4.20b}$$

and the stress resultants of the form

$$T^{\alpha i}_{b(n)} = T^{\alpha i}_{(n)} - b^\alpha_\beta T^{\beta\sigma}_{(n+1)}\delta^i_\sigma; \quad M^{\alpha i}_{b(n)} = M^{\alpha i}_{(n)} - b^\alpha_\beta M^{\beta\sigma}_{(n+1)}\delta^i_\sigma \tag{4.21}$$

are defined.

Initial conditions. The prescribed initial functions of the deformation field are represented by the power series expansions as in Eqs. (3.2) and (3.3), and then the initial conditions (2.18) and (2.19) are given by

$$u^{(n)}_i(\theta^\alpha, t_0) - v^{*(n)}_i(\theta^\alpha) = 0; \quad w^{*(n)}_i(\theta^\alpha) - \dot{u}^{(n)}_i(\theta^\alpha, t_0) = 0 \quad \text{on } A(t_0) \tag{4.22}$$

for the displacement components and

$$\phi^{(n)}_i(\theta^\alpha, t_0) - \varphi^{*(n)}_i(\theta^\alpha) = 0; \quad \Phi^{*(n)}_i(\theta^\alpha) - \dot{\phi}^{(n)}_i(\theta^\alpha, t_0) = 0 \quad \text{on } A(t_0) \tag{4.23}$$

for the microrotation components.

5. A higher-order theory of micropolar shells

So far a hierarchical system of the 2-D higher-order equations is derived for the motions of a functionally graded micropolar thin shell of uniform thickness. It is consist of the gradient and divergence equations (3.8) and (4.5), the constitutive relations (3.15), the natural boundary conditions (4.12) and the initial conditions (4.22) and (4.23). All the equations are derived in variational form except the initial conditions in differential form. Even the initial conditions may be derived in variational form through the appropriate dislocation potential reported in (Altay and Dokmeci, 2004a,b). Now, inserting the foregoing equations into the variational Eq. (2.23), the system of the 2-D shear deformable equations of the shell is expressed in variational form as

$$\delta L_S\{\Lambda_S\} = \delta L_{Sn}\{\Lambda_{Sn}\} + \delta L_{Sp}\{\Lambda_{Sp}\} = 0, \quad \Lambda_S = \Lambda_{Sn} \cup \Lambda_{Sp} \tag{5.1a}$$

with the admissible state as

$$\Lambda_{Sn} = \{u^{(n)}_i \in C_{12}, e^{(n)}_{ij} \in C_{00}, T^{ij}_{(n)} \in C_{10}\}, \quad \Lambda_{Sp} = \{\phi^{(n)}_i \in C_{12}, \epsilon^{(n)}_{ij} \in C_{00}, M^{ij}_{(n)} \in C_{10}\} \tag{5.1b}$$

Here, the denotation of the form

$$\begin{aligned} \delta L_{Sn}\{\Lambda_{Sn}\} = &\int_T \mathrm{d}t \int_A \mathrm{d}A \sum_{n=0}^N \left\{ (e^{(n)}_{ij} - \varsigma^{(n)}_{ij})\delta T^{ij}_{(n)} + \left(T^{ij}_{(n)} - \frac{\partial \Gamma}{\partial e_{ij}}\right)\delta e^{(n)}_{ij} + [V^i_{(n)} + \rho(F^i_{(n)} - \ddot{A}^i_{(n)})]\delta u^{(n)}_i \right\} \\ &+ \int_T \mathrm{d}t \int_{C_t} \sum_{n=0}^N [T^i_{*(n)} - v_\alpha(T^{\sigma i}_{(n)} - b^\alpha_\beta T^{\sigma\beta}_{(n+1)})]\delta u^{(n)}_i \,\mathrm{d}c + \int_T \mathrm{d}t \int_{C_u} \sum_{n=0}^N v_\alpha(u^{(n)}_i - u^{*(n)}_i)\delta T^{\alpha i}_{(n)} \,\mathrm{d}c \\ &+ \int_T \mathrm{d}t \int_{S_{\mathrm{uf}}} \sum_{n=0}^N (P^i_{*n(n)} - P^i_{n(n)})\delta u^{(n)}_i \,\mathrm{d}A + \int_T \mathrm{d}t \int_{S_{\mathrm{lf}}} \sum_{n=0}^N (Q^i_{*n(n)} + Q^i_{n(n)})\delta u^{(n)}_i \,\mathrm{d}A \end{aligned} \tag{5.2}$$

$$\delta L_{Sp}\{\Lambda_{Sp}\} = \int_T \mathrm{d}t \int_A \mathrm{d}A \sum_{n=0}^{N} \left\{ (\epsilon_{ij}^{(n)} - \zeta_{ij}^{(n)})\delta M_{(n)}^{ij} + \left(M_{(n)}^{ij} - \frac{\partial \Gamma}{\partial \epsilon_{ij}} \right) \delta \epsilon_{ij}^{(n)} + \left[W_{(n)}^{i} + \rho (L_{(n)}^{i} - \ddot{B}_{(n)}^{i}) \right] \delta \phi_i^{(n)} \right\}$$
$$+ \int_T \mathrm{d}t \int_{C_m} \sum_{n=0}^{N} \left[M_{*(n)}^{i} - v_\alpha (M_{(n)}^{\sigma i} - b_\beta^\alpha M_{(n+1)}^{\sigma\beta}) \right] \delta \phi_i^{(n)} + \int_T \mathrm{d}t \int_{C_\phi} \sum_{n=0}^{N} v_\alpha (\phi_i^{(n)} - \phi_i^{*(n)}) \delta M_{(n)}^{\alpha i} \, \mathrm{d}c$$
$$+ \int_T \mathrm{d}t \int_{C_\phi} \sum_{n=0}^{N} v_\alpha (\phi_i^{(n)} - \phi_i^{*(n)}) \delta M_{(n)}^{\alpha i} \, \mathrm{d}c + \int_T \mathrm{d}t \int_{S_{\mathrm{uf}}} \sum_{n=0}^{N} (P_{*p(n)}^{i} - P_{p(n)}^{i}) \delta \phi_i^{(n)} \, \mathrm{d}A$$
$$+ \int_T \mathrm{d}t \int_{S_{\mathrm{lf}}} \sum_{n=0}^{N} (Q_{*p(n)}^{i} + Q_{p(n)}^{i}) \delta \phi_i^{(n)} \, \mathrm{d}A \tag{5.3}$$

is defined. The system of the 2-D equations is derived in invariant form that may be readily expressible in a specific set of coordinates most suitable to the shell geometry.

Next, the variational equation (5.1) leads, as its Euler–Lagrange equations, to the system of the 2-D equations of the micropolar shell in differential form, namely

$$e_{ij}^{(n)} - \varsigma_{ij}^{(n)} = 0 \quad \text{on } A \times T \tag{5.4a}$$
$$T_{(n)}^{ij} - \frac{\partial \Gamma}{\partial e_{ij}^{(n)}} = 0, \quad T_{(n)c}^{ij} - T_{(n)}^{ij} = 0 \quad \text{on } A \times T \tag{5.4b}$$
$$V_{(n)}^{i} + F_{(n)}^{i} - \ddot{A}_{(n)}^{i} = 0 \quad \text{on } A \times T \tag{5.4c}$$

and

$$T_{*(n)}^{i} - v_\alpha (T_{(n)}^{\alpha i} - b_\beta^\alpha T_{(n+1)}^{\sigma\beta} \delta_\sigma^i) = 0 \quad \text{on } C_t \times T \tag{5.5a}$$
$$P_{*n(n)}^{i} - P_{n(n)}^{i} = 0 \quad \text{on } S_{\mathrm{uf}} \times T; \quad Q_{*n(n)}^{i} + Q_{n(n)}^{i} = 0 \quad \text{on } S_{\mathrm{lf}} \times T \tag{5.5b}$$
$$u_i^{(n)} - u_i^{*(n)} = 0 \quad \text{on } C_u \times T \tag{5.5c}$$

and also,

$$\in_{ij}^{(n)} - \zeta_{ij}^{(n)} = 0 \quad \text{on } A \times T \tag{5.6a}$$
$$M_{(n)}^{ij} - \frac{\partial \Gamma}{\partial \epsilon_{ij}^{(n)}} = 0, \quad M_{(n)c}^{ij} - M_{(n)}^{ij} = 0 \quad \text{on } A \times T \tag{5.6b}$$
$$W_{(n)}^{i} + L_{(n)}^{i} - \ddot{B}_{(n)}^{i} = 0 \quad \text{on } A \times T \tag{5.6c}$$

and

$$M_{*(n)}^{i} - v_\alpha (M_{(n)}^{\alpha i} - b_\beta^\alpha M_{(n+1)}^{\sigma\beta} \delta_\sigma^i) = 0 \quad \text{on } C_m \times T \tag{5.7a}$$
$$P_{*p(n)}^{i} - P_{p(n)}^{i} = 0 \quad \text{on } S_{\mathrm{uf}} \times T; \quad Q_{*p(n)}^{i} + Q_{p(n)}^{i} = 0 \quad \text{on } S_{\mathrm{lf}} \times T \tag{5.7b}$$
$$\phi_i^{(n)} - \phi_i^{*(n)} = 0 \quad \text{on } C_\phi \times T \tag{5.7c}$$

in the notation of Eqs. (3.9), (3.10), (4.6), (4.20) and (4.21), since the volumetric and surface variations of the admissible state are arbitrary.

The system of the 2-D equations of the micropolar shell is evidently deterministic with $n \in [0, N]$ where N is a finite number selected for a particular case of interest. It comprises the $(42N)$ equations governing the $(42N)$ dependent variables of the admissible state Λ_S. Moreover, the linear system of the 2-D equations defines the initial-mixed boundary value problems that have a unique solution under some prescribed boundary and initial conditions, as is shown in the next section.

6. Uniqueness of solutions

The classical result dealing with the uniqueness of solutions was credited to Kirchhoff in linear elastostatics and Neumann in linear elastodynamics (e.g., Love, 1944). Both of the results were relied on the positive-definiteness of stored energies that is one of the irrevocable axioms in continuum physics. By appealing to the energy argument, the uniqueness of solutions was investigated in the 3-D linear fundamental equations of certain continua (e.g., Weiner, 1957; Mindlin, 1974; Altay and Dökmeci, 2004b) as well as in the linear 1-D or 2-D linear equations of structural elements (e.g.,Warner, 1965; Green and Naghdi, 1971; Mindlin, 1974; Dokmeci, 1978). Mindlin (1974) enumerated the conditions sufficient for the uniqueness in solutions of both the 3-D fundamental equations and the 2-D equations of high frequency equations of crystal and thermopiezoelectric plates. Green and Naghdi (1971) were concerned with the uniqueness in the linear theory of elastic plates and shells, as did Dokmeci (1978) in the theory of thermopiezoelectric laminae. Apart from the energy argument, the methods of analyticity and logarithmic convexity argument, those involving the reflection principles and the Protter technique, and Holmgren's theorem are currently available in investigating the uniqueness of solutions (Knops and Payne, 1972). The methods with their own features take a mathematical argument as fundamental, whereas the energy argument is based upon a physical axiom, and it is of wide use in the literature due to its relative simplicity (e.g., Altay and Dokmeci, 2001). Nevertheless, the newer logarithmic convexity argument (e.g., Altay and Dökmeci, 1998) that does not impose a definiteness condition on the material elasticities is preferred for its own intrinsic interest in proving the following theorem of uniqueness in solutions of the 2-D fully linear theory of the micropolar homogeneous shell.

Theorem. With reference to the θ^i-set of geodesic normal coordinates in the Euclidean 3-D space Ξ, consider the micropolar shell of uniform thickness with its regular region $V + \partial V$, boundary surface $\partial V(=S_e \cup S_{\mathrm{lf}} \cup S_{\mathrm{uf}})$, closure $\overline{V}(= V \cup \partial V)$ and the middle surface A at the time interval $T = [t_0, t_1)$ under a prescribed initial data. The shell region is set in a motion that is maintained by an application of the assigned surface tractions and couples and also the prescribed deformation field over an appropriate portions of the boundary surface S. Now, let

$$\Lambda_S = \Lambda_n \cup \Lambda_p; \quad \Lambda_n = \{u_i^{(n)}, e_i^{(n)}, T^{ij}_{(n)}\}, \quad \Lambda_p = \{\phi_i^{(n)}, \epsilon_i^{(n)}, M^{ij}_{(n)}\} \quad \text{on } A \times T \tag{6.1}$$

be an admissible state of the functions of (θ^α and t) that satisfies the hierarchical system of the 2-D gradient equations, Eqs. (5.4a) and (5.6a), linear constitutive relations, Eqs. (5.4b) and (5.6b) together with Eqs. (3.16) and (3.17), divergence equations, Eqs. (5.4c) and (5.6c), boundary conditions, Eqs. (5.5) and (5.7), and initial conditions (4.22) and (4.23). Also, let the mass density and the microinertia are positive definite on the middle surface A of the shell. Then, there exists at most one admissible state Λ_S of Eq. (6.1) that satisfies the aforementioned equations of the micropolar shell.

In order to prove the theorem, one considers, as usual, the existence of two possible sets of the admissible state, $\Lambda_S^{(n)\alpha}$, that is, two sets of the three displacements, nine strains and nine stress resultants of order (n) and the three microrotations, nine microstrains and nine couple stress resultants of order (n) that satisfy the six divergence equations, eighteen gradient equations and eighteen constitutive relations of order (n). Let the difference admissible state be in the form

$$\begin{aligned}\Lambda_S^{(n)} = \Lambda_S^{(n)1} - \Lambda_S^{(n)2} = \{&u_i^{(n)}(= u_i^{(n)2} - u_i^{(n)1}), \quad e_{ij}^{(n)}(= e_{ij}^{(n)2} - e_{ij}^{(n)1}), \quad T^{ij}_{(n)}(= T^{ij}_{(n)2} - T^{ij}_{(n)1}); \\ &\phi_i^{(n)}(= \phi_i^{(n)2} - \phi_i^{(n)1}), \quad \epsilon_{ij}^{(n)}(= \epsilon_{ij}^{(n)2} - \epsilon_{ij}^{(n)1}), \quad M^{ij}_{(n)}(= M^{ij}_{(n)2} - M^{ij}_{(n)1})\}\end{aligned} \tag{6.2}$$

By virtue of the its linearity, the difference admissible state $\Lambda_S^{(n)}$ evidently satisfies the homogeneous system of the 2-D equations of the micropolar shell. The homogeneous system of the 2-D shell equations consists of the gradient Eqs. (5.4a) and (5.6a), the linear constitutive relations (5.4b) and (5.6b) with Eqs. (3.16) and (3.17), the divergence equations from Eqs. (5.4c) and (5.6c) as

$$V^i_{(n)} - \ddot{A}^i_{(n)} = 0; \quad W^i_{(n)} - \ddot{B}^i_{(n)} = 0 \tag{6.3}$$

the boundary conditions from Eqs. (5.5) and (5.7) as

$$\begin{aligned} &\nu_\alpha(T^{\sigma i}_{(n)} - b^\alpha_\beta T^{\sigma\beta}_{(n+1)}\delta^i_\sigma) = 0, \quad \text{on } C_t \times T; \\ &P^i_{n(n)} = 0 \quad \text{on } S_{\text{uf}} \times T; \quad Q^i_{n(n)} = 0 \quad \text{on } S_{\text{lf}} \times T \\ &u^{(n)}_i = 0 \quad \text{on } C_u \times T \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} &\nu_\alpha(M^{\sigma i}_{(n)} - b^\alpha_\beta M^{\sigma\beta}_{(n+1)}\delta^i_\sigma) = 0 \quad \text{on } C_m \times T \\ &P^i_{p(n)} = 0 \quad \text{on } S_{\text{uf}} \times T; \quad Q^i_{p(n)} = 0 \quad \text{on } S_{\text{lf}} \times T \\ &\phi^{(n)}_i = 0 \quad \text{on } C_\phi \times T \end{aligned} \tag{6.5}$$

and the initial conditions from Eqs. (4.22) and (4.23) as

$$\begin{aligned} &u^{(n)}_i(\theta^\alpha, t_0) = 0; \quad \dot{u}^{(n)}_i(\theta^\alpha, t_0) = 0 \quad \text{on } A(t_0) \\ &\phi^{(n)}_i(\theta^\alpha, t_0) = 0; \quad \dot{\phi}^{(n)}_i(\theta^\alpha, t_0) = 0 \quad \text{on } A(t_0) \end{aligned} \tag{6.6}$$

Moreover, the total energy E_S of the micropolar shell in terms of the energy densities e, k and Σ as

$$E_S = \int_V e\,\mathrm{d}V = \int_A \mathrm{d}A \int_Z (k + \Sigma)\mu\,\mathrm{d}z \tag{6.7}$$

is recorded and the potential energy density Σ from Eq. (2.7) and the kinetic energy density from Eq. (2.22). Thus, one reads for the total energies of the shell as

$$K_S = \frac{1}{2}\int_A \mathrm{d}A \int_Z \rho(\dot{u}^i\dot{u}_i + J^{ij}\dot{\phi}_i\dot{\phi}_j)\mu\,\mathrm{d}z \tag{6.8}$$

$$U_S = \frac{1}{2}\int_A \mathrm{d}A \int_Z (t^{ij}e_{ij} + m^{ij}\epsilon_{ij})\mu\,\mathrm{d}z \tag{6.9}$$

for the difference state Λ_S in Eq. (6.2). Expressing the deformation components by the shifted ones through Eq. (A.10), and then substituting the series expansions (3.2) and (3.3) into Eq. (6.6), and integrating with respect to the thickness coordinate, the kinetic energy of the shell is obtained as

$$K_S = \frac{1}{2}\int_A \mathrm{d}A \sum_{n=0}^{N} (\dot{A}^i_{(n)}\dot{u}^i_{(n)} + \dot{B}^i_{(n)}\dot{\phi}^i_{(n)}) \tag{6.10}$$

where the displacement and microrotation of order (n) in Eq. (4.9) are considered.

Likewise, inserting Eqs. (3.9) and (3.10) into Eq. (6.9), integrating with respect to z and applying the Green–Gauss theorem for the regular region of the shell, the potential energy is expressed by

$$\begin{aligned} U_S = B_f + \frac{1}{2}\int_A \mathrm{d}A \sum_{n=0}^{N} \Big\{ &[-(T^{\beta\alpha}_{(n)} - b^\alpha_\sigma T^{\beta\sigma}_{(n+1)})_{:\,\beta} + b^\alpha_\beta T^{\beta 3}_{(n)} - n(T^{3\alpha}_{(n-1)} - b^\alpha_\beta T^{3\beta}_{(n)})]u^{(n)}_\alpha \\ &+ [-T^{\alpha 3}_{(n):\,\alpha} - b_{\alpha\beta}(T^{\beta\alpha}_{(n)} - b^\alpha_\sigma T^{\beta\sigma}_{(n+1)}) + n(T^{33}_{(n-1)})]u^{(n)}_3 + [-(M^{\beta\alpha}_{(n)} - b^\alpha_\sigma M^{\beta\sigma}_{(n+1)})_{:\,\beta} + b^\alpha_\beta M^{\beta 3}_{(n)} \\ &- n(M^{3\alpha}_{(n-1)} - b^\alpha_\beta M^{3\beta}_{(n)})]\phi^{(n)}_\alpha + [-M^{\alpha 3}_{(n):\,\alpha} - b_{\alpha\beta}(M^{\beta\alpha}_{(n)} - b^\alpha_\sigma M^{\beta\sigma}_{(n+1)}) + n(M^{33}_{(n-1)})]\phi^{(n)}_3 \\ &- \bar{\varepsilon}_{\beta\sigma}(a^{\alpha\beta}T^{\sigma[3]}_{(n)} + d^{\beta\alpha}T^{\sigma[3]}_{(n+1)})\phi^{(n)}_\alpha - \bar{\varepsilon}_{\beta\sigma}(T^{\alpha\beta}_{(n)} - b^\sigma_\sigma T^{\alpha\beta}_{(n+1)} + R_{\mathrm{g}} T^{\alpha\beta}_{(n+2)})\phi^{(n)}_3 \Big\} \end{aligned} \tag{6.11}$$

with

$$B_f = \frac{1}{2}\oint_C \mathrm{d}c \sum_{n=0}^{N} v_\alpha[(T^{\sigma\alpha}_{(n)} - b^\alpha_\beta T^{\beta\sigma}_{(n+1)})u^{(n)}_\alpha + T^{\alpha 3}_{(n+1)}u^{(n)}_3 + (M^{\sigma\alpha}_{(n)} - b^\alpha_\beta M^{\beta\sigma}_{(n+1)})\phi^{(n)}_\alpha + M^{\alpha 3}_{(n+1)}\phi^{(n)}_3] \tag{6.12}$$

in terms of the stress and couple stress resultants (3.11). On comparison of Eq. (6.12) with Eqs. (4.6) and (4.7) and use of Eq. (6.3) for the acceleration terms, the potential energy is given by

$$U_S = B_e + B_f - \frac{1}{2}\int_A \mathrm{d}A \sum_{n=0}^{N}(\ddot{A}^i_{(n)}u^i_{(n)} + \ddot{B}^i_{(n)}\phi^i_{(n)}) \tag{6.13}$$

with

$$B_e = \frac{1}{2}\int_A \mathrm{d}A \sum_{n=0}^{N}[(P^i_{n(n)} - Q^i_{n(n)})u^{(n)}_i + (P^i_{p(n)} - Q^i_{p(n)})\phi^{(n)}_i] \tag{6.14}$$

A substitution of Eqs. (6.10) and (6.13) into Eq. (6.7) yields the total energy as

$$E_S = B_e + B_f + \frac{1}{2}\int_A \mathrm{d}A \sum_{n=0}^{N}\left(\dot{A}^i_{(n)}\dot{u}^{(n)}_i + \dot{B}^i_{(n)}\dot{\phi}^{(n)}_i - \ddot{A}^i_{(n)}u^{(n)}_i - \ddot{B}^i_{(n)}\phi^{(n)}_i\right) \tag{6.15}$$

for the difference set of solutions Λ_S. The stored energy density e is non-negative, by definition and initially zero; so that the total energy E_S of the micropolar shell calculated from the difference state have the same properties.

Now, for the uniqueness of solutions, a logarithmic function (Knops and Payne, 1972) of the form

$$G(t) = \log H(t), \quad t_0 \leqslant t_1 < t < t_2 \leqslant \tau \tag{6.16}$$

with

$$H(t) = \frac{1}{2}\int_A \mathrm{d}A \int_Z \rho(u^i u_i + J^{ij}\phi_i\phi_j)\mu\,\mathrm{d}z \quad \text{at } T = [t_0, t_1) \tag{6.17}$$

is introduced for the micropolar shell. The integrand of this function, and hence the function itself is non-negative, since the mass density and the microinertia are positive definite.

Apparently, a trivial solution is implied for the difference state Λ_S by the condition $H \equiv 0$ for all $t \in [0, T]$. Accordingly, without a loss of generality, $H(t) \equiv 0$ is taken at the time interval $T_1 = [t_0, \tau_1)$ and $T_2 = [\tau_2, t_1)$, and also, it is assumed that $H(t) > 0$ at the time interval between them, $\tau_1 < t < \tau_2$. Then, it is shown that $G(t)$ is a convex function in the interval, namely

$$\ddot{G}(t) = \left(\frac{\dot{H}}{H}\right)^{\cdot} = \frac{H\ddot{H} - \dot{H}^2}{H^2} > 0 \tag{6.18a}$$

or equivalently,

$$\chi = H\ddot{H} - \dot{H}^2 > 0 \quad \text{in } \tau_1 < t < \tau_2 \tag{6.18b}$$

since $H(t) > 0$ is considered. To verify the condition (6.18), the first- and second-derivatives of $H(t)$ are calculated as

$$\dot{H}(t) = \int_A \mathrm{d}A \int_Z \rho(\dot{u}^i u_i + J^{ij}\dot{\phi}_i\phi_j)\mu\,\mathrm{d}z \tag{6.19}$$

and

$$\ddot{H}(t) = \int_A \mathrm{d}A \int_Z \rho(\ddot{u}^i u_i + J^{ij}\ddot{\phi}_i\phi_j + \dot{u}^i\dot{u}_i + J^{ij}\dot{\phi}_i\dot{\phi}_j)\mu\,\mathrm{d}z \tag{6.20}$$

where the smoothness of the function is assumed. Replacing the deformation components from Eqs. (3.2) and (3.3) together with Eq. (A.9), integrating with respect to z and recalling the components of order (n) in Eq. (4.10), the function $H(t)$ by

$$H(t) = \frac{1}{2}\int_A \mathrm{d}A \sum_{n=0}^{N}(A^i_{(n)}u_i^{(n)} + B^i_{(n)}\phi_i^{(n)}) \tag{6.21}$$

and its derivatives by

$$\dot{H}(t) = \int_A \mathrm{d}A \sum_{n=0}^{N}(\dot{A}^i_{(n)}u_i^{(n)} + \dot{B}^i_{(n)}\phi_i^{(n)}) \tag{6.22}$$

$$\ddot{H}(t) = \int_A \mathrm{d}A \sum_{n=0}^{N}(\ddot{A}^i_{(n)}u_i^{(n)} + \ddot{B}^i_{(n)}\phi_i^{(n)} + \dot{A}^i_{(n)}\dot{u}_i^{(n)} + \dot{B}^i_{(n)}\dot{\phi}_i^{(n)}) \tag{6.23}$$

are obtained.

As a last step, Eqs. (6.21)–(6.23) are substituted into the convexity condition with the result

$$\chi = \frac{d^2G}{\mathrm{d}t^2} = \int_A \mathrm{d}A \sum_{n=0}^{N}(A^i_{(n)}u_i^{(n)} + B^i_{(n)}\phi_i^{(n)}) \int_A \mathrm{d}A \sum_{n=0}^{N}(\ddot{A}^i_{(n)}u_i^{(n)} + \ddot{B}^i_{(n)}\phi_i^{(n)}) - \left[\int_A \mathrm{d}A \sum_{n=0}^{N}(\dot{A}^i_{(n)}u_i^{(n)} + \dot{B}^i_{(n)}\phi_i^{(n)})\right]^2 \geqslant 0 \tag{6.24}$$

This is readily satisfied by virtue of Schwartz's inequality (e.g., Knops and Steel, 1969), and hence, the convexity is established for the function $G(t)$. By integrating twice with respect to time, the convexity condition becomes

$$G(t) = \log H(t) = C_0 + C_1 t \geqslant 0 \tag{6.25}$$

With the continuity condition of the function $H(t)$ at $t = \tau_1$ and $t = \tau_2$, one obtains

$$H(t) \leqslant [H(\tau_2)]^{(t-t_1)/\tau}[H(\tau_1)]^{(t_1-t)/\tau}; \quad \tau = \tau_2 - \tau_1,\ \tau_1 < t < \tau_2 \tag{6.26}$$

and then, contrary to the hypothesis above

$$H(t) = 0 \quad \text{for } t \in [\tau_1, \tau_2] \tag{6.27}$$

since $H(\tau_1) = 0$ is assumed at the outset. Thus, $H(t) = 0$ in the time interval $\tau_1 \leqslant t \leqslant \tau_2$ and by continuity assumptions of the field variables Λ_S of the micropolar shell, one arrives at the conclusion that the deformation components vanish, that is to say, the difference set of solutions is trivial and $\Lambda_S^{(1)}$ and $\Lambda_S^{(2)}$ are identical, and hence the theorem is proved without imposing the positive-definiteness of material elasticities.

The theorem may be also proved by use of the classical energy argument as was shown in thermoelastic shells (Rubin, 1986) and in thermopiezoelectric shells (Dokmeci, 1978). Moreover, the theorem states only that there exists at most one set of solutions for the micropolar shell. The complementary theorem that states the condition for which there exists at least one set of solutions to the system of the 2-D equations of the micropolar shell remains to be investigated.

7. Conclusions

The main thrust of this paper is to derive consistently a unified theory for the high frequency vibrations of micropolar thin elastic shells in invariant, differential and variational forms. As a first step toward the theory, a generalised variational principle deduced from Hamilton's principle is presented. The variational principle leads, as its Euler–Lagrange equations, to all the 3-D fundamental equations of micropolar elasticity, except the initial conditions. In the second step, the deformation field of a micropolar shell is chosen as a point of departure for the derivation, and it is represented by the power series expansions in the thickness coordinate. Then, the series expansions of the deformation field together with the variational principle are used to deduce in a rational manner the hierarchical deterministic system of the 2-D equations of the functionally graded micropolar thin shell from the 3-D fundamental equations. The 2-D shear deformable shell equations incorporate all the significant non-polar and polar effects, and they govern all the types of the vibrations of the micropolar shell at both low and high frequency. In the last step, the uniqueness is investigated and the conditions sufficient for the uniqueness in solutions of the system of the 2-D equations are enumerated.

The variational principle with its well-known features operating on all the field variables in micropolar elasticity is the counterpart of the generalised Hellinger–Reissner variational principle in elasticity. The system of the 2-D higher order shell equations in invariant form is quite general, has no counterpart in micropolar elasticity, and it is readily expressible in a particular system of coordinates most suitable to the shell geometry. The fully variational form of the 2-D shell equations has the advantage of free and simple choice of the trial functions in the direct computation and simultaneous approximation upon all the field variables. Besides, a theorem of uniqueness is proved in solutions of the system of the 2-D equations by use of the logarithmic convexity argument (i.e., without imposing the positive definiteness of material elasticities) in lieu of the classical energy argument used for elastic shells and plates (cf., Green and Naghdi, 1971). On the other hand, certain cases involving special kinematics, material and geometrical configuration and the alike can be obtained from the system of the shell equations. The order of $N = 1$ is the closest to Love's first and second approximation for the elastic, anisotropic and functionally graded micropolar shells. If the shell material is isotropic, the constitutive relations follow from Eq. (2.11) and Eqs. (3.16) and (3.17). In the case of plates, the curvature effects disappears, that is, $b_{\alpha\beta} = 0$ and the shell tensor becomes the Kronecker deltas, $\mu^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$ and $\mu = 1$ in Eqs. (A.3b), (A.4)–(A.15), and the approximation of order $N = 1$ recovers a generalized version of the 2-D equations of micropolar plates reported by Eringen (1967). For a micropolar thin beam of rectangular cross sections or a micropolar strip, the dependency on the width coordinate x_2 is eliminated, and the deformation field degenerates into $u_i^{(n)}(x_\alpha, t) = u_i^{(n)}(x_1, t)$ and $\phi_i^{(n)}(x_\alpha, t) = \phi_i^{(n)}(x_1, t)$. In such a case, the system of equations for the approximation of order $N = 1$ leads to the Bernoulli–Euler and Timoshenko equations of micropolar beams (cf., Dökmeci, 1975). Moreover, omitting all the micropolar terms, one recovers the higher order theories of functionally graded elastic non-polar thin shells and plates.

Lastly, the existence, stability and error estimates of solutions and, in particular, the inclusion of the shell parameter into the system of the 2-D equations of the micropolar shell remain to be investigated. Nevertheless, the resulting equations appear to have many interesting features that will be focused on the follow-up papers. The next step is to consider some applications as well as extensions of the system of 2-D equations involving with time and/or temperature dependent continua.

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Appendix A. Preliminaries of a surface geometry

In the Euclidean 3-D space Ξ, certain results from the differential geometry of a surface are summarized to be self-contained in the derivation of the 2-D equations of micropolar shells, a more elaborate account of which are given (e.g., Librescu, 1975).

Metric properties. Let $\theta^i = \{\theta^\alpha, \theta^3 \equiv z\}$ be identified with a set of geodesic normal coordinates and let the regular region of a shell space $V + \partial V$, with its boundary surface $\partial V\,(=S_e \cup S_{\mathrm{lf}} \cup S_{\mathrm{uf}})$ be referred to the set of coordinates. Also, let $z = 0$, $z = -h$ and $z = h$ define the middle surface (or reference surface) A, the lower face S_{lf} and upper face S_{uf} of the shell, respectively. The edge boundary surface S_e is a right cylindrical surface and intersects the reference surface A along a Jordan curve C. The θ^α-coordinate curves form a set of curvilinear coordinates on the reference surface. With reference to the θ^i-set of normal coordinates, the position vector $\mathbf{R}$ of a generic point P in the shell region takes the form

$$\mathbf{R}(\theta^i) = \mathbf{r}(\theta^\alpha) + z\mathbf{a}_3(\theta^\alpha) \tag{A.1}$$

subject to the restriction of the form

$$\mathbf{a}_3 \cdot \mathbf{a}_3 = \mathbf{1}, \quad \mathbf{a}_3 \cdot \mathbf{R}_{,\alpha}(\theta^\alpha) = \mathbf{a}_3 \cdot \mathbf{a}_\alpha = \mathbf{a}_{3\alpha} = \mathbf{0} \tag{A.2}$$

Here, $\mathbf{r}$ stands for the position vector of the projection of the point P on, $\mathbf{a}_\alpha$ for the tangential base vectors, and $\mathbf{a}_3$ for the unit vector perpendicular to, the reference surface A. Thus, the base vectors $(\mathbf{g}^i, \mathbf{g}_i)$ and the metric and conjugate tensors (g_{ij}, g^{ij}) of the shell region are given by

$$g_\alpha = a_\alpha + z a_{3,\alpha} = \mu^\beta_\alpha a_\beta, \quad \mathbf{g}^\alpha = (\mu^{-1})^\alpha_\beta a^\beta, \quad \mathbf{g}_3 = \mathbf{g}^3 = \mathbf{a}_3 = \mathbf{a}^3 \tag{A.3a}$$

and

$$g_{\alpha\beta} = \mu^\nu_\alpha \mu^\sigma_\beta a_{\nu\sigma}, \quad g_{\alpha 3} = 0, \quad g_{33} = 1; \quad g^{\alpha\beta} = (\mu^{-1})^\alpha_\nu (\mu^{-1})^\beta_\sigma a^{\nu\sigma}, \quad g^{\alpha 3} = 0, \quad g^{33} = 1 \tag{A.3b}$$

in terms of those defined for the reference surface A of the form

$$\begin{aligned} &\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} = \mathbf{g}_\alpha(\theta^\beta, 0); \quad \mathbf{a}_\alpha \cdot \mathbf{a}_\beta = a_{\alpha\beta} = g_{\alpha\beta}(\theta^\sigma, 0), \quad \mathbf{a}_3 \cdot \mathbf{a}_3 = a_{33} = 1 \\ &\mathbf{a}^\alpha . \mathbf{a}^\beta = a^{\alpha\beta} = g^{\alpha\beta}(\theta^\sigma, 0), \quad \mathbf{a}^\alpha \cdot \mathbf{a}^3 = a^{\alpha 3} = 0, \quad a^{\alpha\sigma} a_{\sigma\beta} = \delta^\alpha_\beta \end{aligned} \tag{A.4}$$

Here, μ^α_β or $(\mu^{-1})^\alpha_\beta$ denotes the shell tensor that plays the role of shifters between the space and surface tensors, and it is defined by

$$\mu^\alpha_\beta = \delta^\alpha_\beta - z b^\alpha_\beta, \quad \mu^\alpha_\sigma (\mu^{-1})^\sigma_\beta = \delta^\alpha_\beta \tag{A.5}$$

with the relations of the form

$$\begin{aligned} &\mu = |\mu^\alpha_\beta| = (g/a)^{1/2} = 1 - 2zR_{\mathrm{m}} + z^2 R_{\mathrm{g}}; \quad g = |g_{ij}|, \quad a = |a_{\alpha\beta}| \\ &\mu(\mu^{-1})^\alpha_\beta = \delta^\alpha_\beta + z(b^\alpha_\beta - b^\sigma_\sigma \delta^\alpha_\beta) = \delta^\alpha_\beta + z d^\alpha_\beta; \quad d^\alpha_\beta = b^\alpha_\beta - b^\sigma_\sigma \delta^\alpha_\beta; \quad a^{\lambda\alpha} d^\beta_\lambda = d^{\alpha\beta} \end{aligned} \tag{A.6}$$

The mean curvature R_{m} and the Gaussian curvature R_{g} of the reference surface A are given by

$$R_{\mathrm{m}} = \tfrac{1}{2} b^\alpha_\alpha, \quad R_{\mathrm{g}} = |b^\alpha_\beta| \tag{A.7}$$

In the above equations, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ denote, respectively, the first and second fundamental form of the reference surface. The third fundamental form is given by $c_{\alpha\beta} = b_{\alpha\sigma} b^\sigma_\beta$.

Relationships between space and surface tensors. A vector $\boldsymbol{\chi}$ may be expressed in terms of either the base vectors $(g_i, \mathbf{g}^i)$ of the shell space V or those $(\mathbf{a}_i, \mathbf{a}^i)$ of the reference surface A, namely

$$\boldsymbol{\chi} = \chi_i \mathbf{g}^i = \chi^i \mathbf{g}_i = \bar{\chi}_\alpha \mathbf{a}^\alpha + \bar{\chi}_3 \mathbf{a}^3 = \bar{\chi}^\alpha \mathbf{a}_\alpha + \bar{\chi}^3 \mathbf{a}_3 \tag{A.8}$$

The shifted components $(\bar{\chi}_i, \bar{\chi}^i)$ of the vector $\boldsymbol{\chi}$ indicated by an overbar are associated with the components (χ_i, χ^i) by

$$\chi_\alpha = \mu^\beta_\alpha \bar{\chi}_\beta,\ \ \chi^\alpha = (\mu^{-1})^\alpha_\beta \bar{\chi}^\beta; \quad \bar{\chi}_\alpha = (\mu^{-1})^\beta_\alpha \chi_\beta, \bar{\chi}^\alpha = \mu^\alpha_\beta \chi^\beta; \quad \chi^3 = \chi_3 = \bar{\chi}^3 = \bar{\chi}_3 \tag{A.9}$$

Moreover, the ε-alternating tensor for the reference surface is defined by

$$\bar{\varepsilon}_{\alpha\beta} = \varepsilon_{\alpha\beta 3}(z=0) = e_{\alpha\beta 3}\sqrt{a},\ \ \bar{\varepsilon}^{\alpha\beta} = \varepsilon^{\alpha\beta 3}(z=0) = e^{\alpha\beta 3}/\sqrt{a}; \quad \varepsilon_{\alpha\beta} = \mu\bar{\varepsilon}_{\alpha\beta},\ \ \varepsilon^{\alpha\beta} = \mu^{-1}\bar{\varepsilon}^{\alpha\beta} \tag{A.10}$$

where $(\bar{\varepsilon}_{\alpha\beta}$ or $\bar{\varepsilon}^{\alpha\beta})$ is the absolute surface tensors, $(\varepsilon_{ijk}$ or $\varepsilon^{ijk})$ is the alternating tensor and $(e_{ijk}$ or $e^{ijk})$ denotes the usual alternating symbol.

Certain useful identities of the form

$$\begin{aligned}
&\chi_{\alpha;\beta} = \mu^\nu_\alpha(\bar{\chi}_{\nu:\beta} - b_{\nu\beta}\bar{\chi}^3), \quad \chi^\alpha_{;\beta} = (\mu^{-1})^\alpha_\nu(\bar{\chi}^\nu_{:\ \beta} - b^\nu_\beta\bar{\chi}^3)\\
&\chi_{\alpha;3} = \mu^\nu_\alpha \bar{\chi}_{\nu,3}, \quad \chi_{3;\alpha} = \bar{\chi}_{3,\alpha} + b^\beta_\alpha \bar{\chi}_\beta, \quad \chi^\alpha_{;3} = (\mu^{-1})^\alpha_\beta \bar{\chi}^\beta_{,3}, \quad \chi^3_{;\alpha} = \bar{\chi}^3_{,\alpha} + b_{\alpha\beta}\bar{\chi}^\beta\\
&\chi^3_{;3} = \chi_{3;3} = \chi_{3,3} = \bar{\chi}_{3,3} = \bar{\chi}^3_{,3}
\end{aligned} \tag{A.11}$$

and those

$$\begin{aligned}
&\mu\mu^\nu_\alpha \chi^{\alpha\beta}_{;\beta} = (\mu\mu^\nu_\eta \chi^{\eta\beta})_{:\ \beta} - \mu\mu^\nu_\alpha(\mu^{-1})^\beta_\lambda b^\lambda_\beta \chi^{\alpha 3} - \mu b^\nu_\beta \chi^{3\beta}\\
&\mu\chi^{3\alpha}_{;\alpha} = (\mu\chi^{3\alpha})_{:\ \alpha} + \mu\mu^\nu_\alpha b_{\nu\beta}\chi^{\alpha\beta} - \mu(\mu^{-1})^\alpha_\nu b^\nu_\beta \chi^{33}\\
&\mu^\beta_\alpha \chi^{\alpha 3}_{;3} = (\mu^\beta_\alpha \chi^{\alpha 3})_{,3}
\end{aligned} \tag{A.12}$$

for the components of an asymmetric tensor χ^{ij}, and that

$$\mu\chi^\alpha_{;\alpha} = (\mu\chi^\alpha)_{:\ \alpha} - \mu(\mu^{-1})^\alpha_\beta b^\beta_\alpha \chi^3 \tag{A.13}$$

for a vector field χ^i, and relation of the form

$$\mu_{,3} = -\mu(\mu^{-1})^\alpha_\beta b^\beta_\alpha \tag{A.14}$$

are recorded, and they are used in replacing the covariant derivatives with respect to the space metrics by those with respect to the surface metrics.

Differential elements. The surface element $\mathrm{d}S$ on the faces, the area element $\mathrm{d}A$ on the reference surface, associated with the volume element $\mathrm{d}V$, and the line element along a Jordan curve C are given in the form

$$\mathrm{d}V = \sqrt{g}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2\,\mathrm{d}z = \mathrm{d}S\,\mathrm{d}z = \mu\mathrm{d}A\,\mathrm{d}z, \quad \mathrm{d}A = \sqrt{a}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2, \quad n_\alpha\mathrm{d}S = \mu\nu_\alpha\,\mathrm{d}c\,\mathrm{d}z \tag{A.15}$$

where, n_i and ν_i are the unit vectors normal to the boundary surface ∂V and the edge surface S_e of the shell.

Appendix B

Nomenclature

Ξ Euclidean 3-D space
$\theta^i(\theta_\alpha, \theta^3 \equiv z)$ fixed, right-handed system of geodesic normal coordinates
$t, T[t_0, t_1)$ time, time interval
$k, \Sigma; K, U$ kinetic and potential energy densities, total kinetic and potential energies
L Lagrangian function
$\delta^* W$ virtual work
Λ admissible state

Ω, $\partial\Omega$, $\overline{\Omega}$ regular micropolar region, its boundary surface and closure
e, E stored energy density and total energy
n_i, v_i unit outward vectors normal to $\partial\Omega$ and S
t^{ij}, m^{ij} stress and couple stress tensors
$\rho; f^i, l^i$ mass density; body force and body couple vectors
u^i, $a^i (= \ddot{u}^i)$ displacement and acceleration vectors
ϕ_i, b^j, J^{ij} microrotation and microacceleration vectors ($=J^{ij}\ddot{\phi}_i$), microinertia tensor
$\bar{u}_i$, $\bar{\phi}_i$ shifted components of the displacement and microrotation vectors
Θ prescribed steady temperature increment
e_{ij}, ϵ_{ij} linear strain and microstrain tensors
$B^{ij}, C^{ijkl}, D^{ijkl}$ material constants
t^i, m^i stress and couple stress vectors
ε^{ijk} alternating tensor
$2h, Z = [-h, h]$ uniform thickness of micropolar shell, thickness interval
R_0 least principal radius of curvature of the shell middle surface
$\epsilon_s = 2h/R_0$ shell parameter
$V, \partial V$ shell region V with its boundary surface ∂V
A, C middle (reference) surface of the shell and a Jordan curve that bounds A
$\mathrm{d}c, \mathrm{d}A, \mathrm{d}S$ line element along C, and area element on the middle surface A and on the upper and lower faces of the shell
$\chi^{ij}_{(n)}$ quantity χ^{ij} of order (n)

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